EXTENDED COHERENT STATES AND MODIFIED PERTURBATION THEORY

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Abstract

An extended coherent state (ECS) for describing a system of two interacting quantum objects is considered. A modified perturbation theory based on using the ECSs is formulated.

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1. Extended coherent states

Coherent states were constructed first by Schrödinger [1] and in the last 40 years of the 20th century were widely used in different problems of quantum physics [2]. There are many modifications of coherent states. Recall, for example, the spin coherent states introduced in [3,4]. A general algebraic approach in the coherent state theory was developed in [4]. The coherent states for a particle on a sphere were applied in [5] to describe the rotator time evolution. Here we propose one more generalization of the theory by introducing the extended coherent state (ECS).

Consider a system of an oscillator and a free spinless particle posessing a momentum $\mathbf{k}_0$. Let $\widehat{b}^\dagger$ and $\widehat{b}$ be the ladder operators for the oscillator. Introduce the creation $\widehat{a}^\dagger$ and annihilation $\widehat{a}$ operators of Bose type to describe a possible change in the particle's state (note that the further

consideration may be applied just as well to a Fermi particle). Input the operator

$$\widehat{Q} = \sum_{\mathbf{q}} h_{\mathbf{q}} \widehat{\rho}_{\mathbf{q}}, \tag{1}$$

where

$$\widehat{\rho}_{\mathbf{q}} = \sum_{\mathbf{k}} \widehat{a}^{\dagger}_{\mathbf{k}} \widehat{a}_{\mathbf{k}+\mathbf{q}}$$

is the Fourier component of the density operator and $h_{\mathbf{q}}$ - are coefficients depending on momentum $\mathbf{q}$. We can construct another linear combination $\widehat{Q}'$ of operators $\widehat{\rho}_{\mathbf{q}}$ with the help of any other set of coefficients $h'_{\mathbf{q}}$. All these combinations are commutative

$$[\widehat{Q}, \widehat{Q}']_{-} = 0$$

because the commutation rule

$$[\widehat{\rho}_{\mathbf{q}}, \widehat{\rho}_{\mathbf{q}'}]_{-} = 0 \tag{2}$$

is fullfilled for all $\mathbf{q}$ and $\mathbf{q}'$.

Input a vector of state $|0, \mathbf{k}_0)$, where the first argument (0) denotes a ground state of the oscillator and the second one $(\mathbf{k}_0)$ describes a state of the particle. Define the vector

$$|h, \mathbf{k}_0 >= \exp\left(-\frac{1}{2}\widehat{Q}^{\dagger}\widehat{Q}\right) \sum_{n=0}^{\infty} \frac{1}{n!} (\widehat{Q}\widehat{b}^{\dagger})^n |0, \mathbf{k}_0) \tag{3}$$

as an ESC (here we briefly denote by h the whole set of coefficients $h_{\mathbf{q}}$). Obviously, the vector (3) coincides with the ordinary Schrödinger coherent state (SCS), when one replaces all particle's operators by their classical equivalents. The ECS describes some state of a system of two interacting quantum objects — the particle and the oscillator. By this circumstance the ECS sufficiently differs from the SCS.

We outline the following general properties of the ECS:

(1) The ECS is not the eigenvector for $\widehat{b}$, but

$$\widehat{b}\, |h, \mathbf{k}_0 >= \widehat{Q} |h, \mathbf{k_0} > . \tag{4}$$

(2) The operators $\widehat{\rho}_{\mathbf{q}}$ only change momenta for all the one-particle states. Hence, the following relations are fullfilled:

$$\widehat{\rho}_{\mathbf{q}} |h, \mathbf{k}_0 >= |h, \mathbf{k}_0 - \mathbf{q} > \tag{5}$$

$$\widehat{\rho}^{\dagger}_{\mathbf{q}}\widehat{\rho}_{\mathbf{q}}|h,\mathbf{k}_0>=|h,\mathbf{k}_0>. \tag{6}$$

(3) There is the following representation:

$$|h,\mathbf{k}_0>=\exp[\widehat{Q}\widehat{b}^{\dagger}-\widehat{Q}^{\dagger}\widehat{b}]|0,\mathbf{k}_0) \tag{7}$$

which is equivalent to the relevant representation of the SCS.

(4) If $h_{\mathbf{q}}=g\,\Delta(\mathbf{q}-\mathbf{q}_0)$ we easily have

$$|h,\mathbf{k}_0>=\exp\left[\frac{-|g|^2}{2}\right]\sum_{n=0}^{\infty}\frac{g^n}{n!}(\widehat{\rho}_{\mathbf{q}_0}\widehat{b}^{\dagger})^n|0,\mathbf{k}_0) \tag{8}$$

and therefore,

$$<h,\mathbf{k}_0|h',\mathbf{k}'_0>=\exp\Big[-\frac{1}{2}(|g|^2+|g'|^2-2g^*g')\Big]\Delta(\mathbf{k}_0-\mathbf{k}'_0). \tag{9}$$

(5) The total amount of ECS are more than sufficient to define the Hilbert space. Following to Klauder [6] (see also [7]) we can introduce the development of the unity operator

$$\widehat{\mathrm{I}}=\sum_{\mathbf{k}}\frac{1}{\pi}\int d^2z\,\widehat{Q}\;|zh,\mathbf{k}><zh,\mathbf{k}|\;\widehat{Q}^{\dagger} \tag{10}$$

where z - is the complex variable, $d^2z=d[\mathrm{Re}(z)]d[\mathrm{Im}(z)]$. To prove the last equation one may use the integral

$$\int d^2z\,(z^*)^n z^m\;\exp\Big[-|z|^2\widehat{Q}^{\dagger}\widehat{Q}\Big]\;\widehat{Q}^{m+1}(\widehat{Q}^{\dagger})^{n+1}=\pi\,n!\,\delta_{nm}.$$

(6) There is the following useful sum rule:

$$\sum_{\mathbf{k}}e^{i\mathbf{s}\mathbf{k}}\widehat{a}_{\mathbf{k}}|h,\mathbf{k_0}>=e^{i\mathbf{s}\mathbf{k}_0}|\alpha)\otimes|\mathrm{vac}_p) \tag{11}$$

where the right-hand side contains a direct product of the SCS for the oscillator

$$|\alpha)=\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{0}^{\infty}\frac{\alpha^n}{n!}(b^{\dagger})^n|0)$$

and a vacuum state of the particle $|\mathrm{vac}_p)$. Here the quantity α is given by the formula

$$\alpha=\sum_{\mathbf{q}}h_{\mathbf{q}}e^{-is\mathbf{q}}.$$

To prove the property (6) one should keep in mind the relation:

$$\sum_{\mathbf{k}}\widehat{a}_{\mathbf{k}}e^{i\mathbf{k}\mathbf{x}}\widehat{\rho}_{\mathbf{q}_1}\widehat{\rho}_{\mathbf{q}_2}...|0,\mathbf{k}_0)=e^{i\mathbf{k}_0\mathbf{x}}e^{-i\mathbf{q}_1\mathbf{x}}e^{-i\mathbf{q}_2\mathbf{x}}...|0)\otimes|vac_p) \tag{12}$$

where $|0)$ is the vector of the ground state of the oscillator.

2. Modified perturbation theory

ECSs, first introduced in 1983[8][1] arise, for example, in a problem of interaction between a moving particle and an oscillator. The proper Hamiltonian can be represented in the following general form

$$\widehat{H}_{int} = \widehat{b}^{\dagger} \sum_{\mathbf{q}} g_{\mathbf{q}} \widehat{\rho}_{\mathbf{q}} + \widehat{b} \sum_{\mathbf{q}} g^{*}_{\mathbf{q}} \widehat{\rho}^{\dagger}_{\mathbf{q}} \tag{13}$$

where $g_{\mathbf{q}}$ - is a coupling function. Since $\widehat{\rho}^{\dagger}_{\mathbf{q}} = \widehat{\rho}_{-\mathbf{q}}$, it should be $g_{-\mathbf{q}} = g^{*}_{\mathbf{q}}$.

In most applications the Hamiltonian (13) within the interaction picture depends on time via the density operators $\widehat{\rho}(t)$. In these cases we cann't apply ECS without some modification of the theory. Indeed, instead of relations (2) we have

$$[\widehat{\rho}_{\mathbf{q}}(t), \widehat{\rho}_{\mathbf{q}'}(t')]_{-} = \sum_{\mathbf{k}} \widehat{a}^{\dagger}_{\mathbf{k}} \widehat{a}_{\mathbf{k}+\mathbf{q}+\mathbf{q}'} [\exp\{i(\varepsilon_{\mathbf{k}} t - \varepsilon_{\mathbf{k}+\mathbf{q}+\mathbf{q}'} t' - i\varepsilon_{\mathbf{k}+\mathbf{q}}(t-t')\}$$
$$- \exp\{i(\varepsilon_{\mathbf{k}} t' - \varepsilon_{\mathbf{k}+\mathbf{q}+\mathbf{q}'} t + i\varepsilon_{\mathbf{k}+\mathbf{q}'}(t-t')\}].$$

We construct a modified perturbation theory with the help of excluding an integrable part of the interaction. For this purpose we expand the operator $\widehat{H}_{int}(t)$ in two parts, $\widehat{H}^{(0)}_{int}(t)$ and $\widehat{H}^{(1)}_{int}(t)$, where

$$\widehat{H}^{(0)}_{int}(t) = \widehat{b}^{\dagger} \sum_{\mathbf{q}} g_{\mathbf{q}} \widehat{\rho}_{\mathbf{q}} f_{\mathbf{q}}(t) + \widehat{b} \sum_{\mathbf{q}} g^{*}_{\mathbf{q}} \widehat{\rho}^{\dagger}_{\mathbf{q}} f^{*}_{\mathbf{q}}(t)$$

$$\widehat{H}^{(1)}_{int}(t) = \widehat{H}_{int}(t) - \widehat{H}^{(0)}_{int}(t).$$

Here the function $f_{\mathbf{q}}(t)$ must be unimodular to preserve the interaction intensity. Obviously, the operators $\sum_{\mathbf{q}} g_{\mathbf{q}} \widehat{\rho}_{\mathbf{q}} f_{\mathbf{q}}(t)$ defined at different times, obey the commutation relations. Then, by virtue of the above consideration, the equation

$$i\frac{d}{dt}|t) = \widehat{H}^{(0)}_{int}(t)|t)$$

acquires an exact solution

$$|t) = e^{-i\widehat{\chi}(t)}|h, \mathbf{k}_0 > \tag{14}$$

where $\widehat{Q}$ has the previous form (1) and

$$h_{\mathbf{q}} = -ig_{\mathbf{q}} \int_{0}^{t} dt' f_{\mathbf{q}}(t')\, e^{i\omega t'}$$

[1] Extended coherent states were first denoted as 'double coherent' states or 'modified coherent' states.

$$\widehat{\chi}(t) = -\frac{i}{2}\int\limits_0^t \{\widehat{\dot{Q}}^\dagger(t')\widehat{Q}(t') - \widehat{Q}^\dagger(t')\widehat{\dot{Q}}(t')\}dt'.$$

The solution (14) can be rewritten as $|t) = \widehat{U}_0(t)|0, \mathbf{k}_0>$, where we introduce a zero-th order evolution operator

$$\widehat{U}_0(t) = \exp\{\widehat{Q}(t)\widehat{b}^\dagger - \widehat{Q}^\dagger(t)\widehat{b} - i\widehat{\chi}(t)\}.$$

There are the following useful commutation relations:

$$[\widehat{b}, \widehat{U}_0(t)]_- = \widehat{U}_0(t)\widehat{Q}(t) \qquad [\widehat{b}, \widehat{U}_0^\dagger(t)]_- = -\widehat{U}_0^\dagger(t)\widehat{Q}(t)$$

$$[\widehat{b}^\dagger, \widehat{U}_0(t)]_- = \widehat{U}_0(t)\widehat{Q}^\dagger(t) \qquad [\widehat{b}^\dagger, \widehat{U}_0^\dagger(t)]_- = -\widehat{U}_0^\dagger(t)\widehat{Q}^\dagger(t).$$

Let us introduce a new representation for the vector of state and for operators:

$$|t>= \widehat{U}_0^\dagger(t)|t) \qquad \widetilde{A} = \widehat{U}_0^\dagger(t)\widehat{A}\widehat{U}_0(t).$$

The new vector of state obeys the equation

$$i\frac{d}{dt}|t>= \widetilde{H}_{int}^{(1)}(t)|t>$$

which can be solved with the help of a standard technique using the T-exponent

$$|t>= \mathrm{Texp}\{-i\int\limits_0^t dt'\widetilde{H}_{int}^{(1)}(t')\}\ |0, \mathbf{k}_0). \tag{15}$$

If the choice of the function $f_{\mathbf{q}}(t)$ ensures the rapid convergence to the series (15), formula (14) gives a good approximation for the vector of state. In this case we can evaluate a wide set of physical characteristics with sufficient accuracy. As an example, we calculate the density matrix for the particle, for which the exact expression is given by the formula

$$\Gamma(\mathbf{x},\, \mathbf{x}',\, t) =< t|\widetilde{\psi}^\dagger(\mathbf{x}, t)\widetilde{\psi}(\mathbf{x}', t)|t> . \tag{16}$$

Here the usual wave operators are introduced, namely,

$$\widetilde{\psi}(\mathbf{x}, t) = \widehat{U}_0^\dagger(t)\widehat{\psi}(\mathbf{x}, t)\widehat{U}_0(t) \qquad \widehat{\psi}(\mathbf{x}, t) = \sum_{\mathbf{k}} \widehat{a}_{\mathbf{k}} \exp\{i\mathbf{k}\mathbf{x} - i\varepsilon_{\mathbf{k}}t\}$$

where $\varepsilon_{\mathbf{k}}$ - is an energy of the particle posessing momentum $\mathbf{k}$. The further consideration will be more convenient if the particle- oscillator interaction began at any incident time $t_0 < 0$ when the oscillator was found in the

ground state. Let us define the density matrix at $t = 0$. In the first approximation we can set $|t> \approx |0, \mathbf{k}_0)$. In this case

$$\Gamma(\mathbf{x},\, \mathbf{x}',\, t) \approx (0, \mathbf{k}_0|\widehat{U}_0^\dagger(0)\widehat{\psi}^\dagger(\mathbf{x}, 0)\widehat{\psi}(\mathbf{x}', 0)\widehat{U}_0(0)|0, \mathbf{k}_0). \tag{17}$$

Using relation (7) we have $\widehat{U}_0(0)|0, \mathbf{k}_0) = e^{-i\widehat{\chi}(0)}|h, \mathbf{k}_0>$, where $\widehat{Q}$ is defined as in (1) with

$$h_{\mathbf{q}} = h_{\mathbf{q}}(0) \qquad h_{\mathbf{q}}(t) = -ig_{\mathbf{q}}\int\limits_{t_0}^{t} f_{\mathbf{q}}(t')e^{i\omega t'}dt' \qquad t > t_0.$$

Now we apply relations (12) to obtain the formula similar to (11):

$$\widehat{\psi}(\mathbf{x}', 0)\widehat{U}_0(t)|0, \mathbf{k}_0) = \exp\{i\mathbf{k}_0\mathbf{x}' - i\Phi(\mathbf{x}')\}|\alpha(\mathbf{x}', 0)) \otimes |vac_p) \tag{18}$$

where

$$\alpha(\mathbf{x}, t) = \sum_{\mathbf{q}} h_{\mathbf{q}}(t)e^{-i\mathbf{q}\mathbf{x}}$$

$$\Phi(\mathbf{x}) = \int\limits_{t_0}^{0} \mathrm{Im}\,[\dot{\alpha}^*(\mathbf{x}, t')\alpha(\mathbf{x}, t')]\ dt'.$$

Substituting (18) into (17) we obtain

$$\Gamma(\mathbf{x},\, \mathbf{x}',\, 0) \approx e^{-i\mathbf{k}_0\mathbf{x}+i\mathbf{k}_0\mathbf{x}'}\times$$

$$\exp\Big\{i\Phi(\mathbf{x}) - i\Phi(\mathbf{x}') - \frac{1}{2}\Big[|\alpha(\mathbf{x}, 0)|^2 + |\alpha(\mathbf{x}', 0)|^2 - 2\alpha^*(\mathbf{x}, 0)\alpha(\mathbf{x}', 0)\Big]\Big\}. \tag{19}$$

Note, that in the case $g_{\mathbf{q}} = g\Delta(\mathbf{q} - \mathbf{q}_0)$, the phase $\Phi(\mathbf{x}) = const$ and formula (19) is simlified.

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